1.- Demostrar

$$\left[\frac{i\,\gamma^{\mu}\gamma^{\nu}}{2},\frac{i\,\gamma^{\rho}\gamma^{\sigma}}{2}\right]=i\left(g^{\nu\rho}\frac{i\,\gamma^{\mu}\gamma^{\sigma}}{2}-g^{\mu\rho}\frac{i\,\gamma^{\nu}\gamma^{\sigma}}{2}-g^{\nu\sigma}\frac{i\,\gamma^{\mu}\gamma^{\rho}}{2}+g^{\mu\sigma}\frac{i\,\gamma^{\nu}\gamma^{\rho}}{2}\right)$$

Consideremos el conmutador

$$\begin{aligned}[\gamma^{\mu}\gamma^{\nu},\gamma^{\rho}] &= \gamma^{\mu}[\gamma^{\nu},\gamma^{\rho}]+[\gamma^{\mu},\gamma^{\rho}]\gamma^{\nu} = \gamma^{\mu}(\gamma^{\nu}\gamma^{\rho}-\gamma^{\rho}\gamma^{\nu})+(\gamma^{\mu}\gamma^{\rho}-\gamma^{\rho}\gamma^{\mu})\gamma^{\nu} = \\ &= \gamma^{\mu}\gamma^{\nu}\gamma^{\rho}-\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}+\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}-\gamma^{\rho}\gamma^{\mu}\gamma^{\nu} = \gamma^{\mu}\gamma^{\nu}\gamma^{\rho}-(2g^{\rho\mu}-\gamma^{\mu}\gamma^{\rho})\gamma^{\nu} = \\ &= \gamma^{\mu}\gamma^{\nu}\gamma^{\rho}-2g^{\rho\mu}\gamma^{\nu}+\gamma^{\mu}\gamma^{\rho}\gamma^{\nu} = \gamma^{\mu}\gamma^{\nu}\gamma^{\rho}-2g^{\rho\mu}\gamma^{\nu}+\gamma^{\mu}(2g^{\rho\nu}-\gamma^{\nu}\gamma^{\rho}) \\ &= \gamma^{\mu}\gamma^{\nu}\gamma^{\rho}-2g^{\rho\mu}\gamma^{\nu}+2g^{\rho\nu}\gamma^{\mu}-\gamma^{\mu}\gamma^{\nu}\gamma^{\rho} = -2g^{\rho\mu}\gamma^{\nu}+2g^{\rho\nu}\gamma^{\mu}\end{aligned}$$

Donde hemos utilizado que $\gamma^{\rho}\gamma^{\mu}+\gamma^{\mu}\gamma^{\rho}=2g^{\rho\mu}$ (1).

Por lo tanto, el conmutador inicial queda

$$\begin{aligned}\left[\frac{i\gamma^{\mu}\gamma^{\nu}}{2},\frac{i\gamma^{\rho}\gamma^{\sigma}}{2}\right] &= \left(\frac{i}{2}\right)^2[\gamma^{\mu}\gamma^{\nu},\gamma^{\rho}\gamma^{\sigma}] = \left(\frac{i}{2}\right)^2(\gamma^{\rho}[\gamma^{\mu}\gamma^{\nu},\gamma^{\sigma}]+[\gamma^{\mu}\gamma^{\nu},\gamma^{\rho}]\gamma^{\sigma}) \\ &= \left(\frac{i}{2}\right)^2(\gamma^{\rho}(-2g^{\sigma\mu}\gamma^{\nu}+2g^{\sigma\nu}\gamma^{\mu})+(-2g^{\rho\mu}\gamma^{\nu}+2g^{\rho\nu}\gamma^{\mu})\gamma^{\sigma}) \\ &= \left(\frac{i}{2}\right)^2(-2g^{\sigma\mu}\gamma^{\rho}\gamma^{\nu}+2g^{\sigma\nu}\gamma^{\rho}\gamma^{\mu}-2g^{\rho\mu}\gamma^{\nu}\gamma^{\sigma}+2g^{\rho\nu}\gamma^{\mu}\gamma^{\sigma}) \\ &= i\left(\boxed{-g^{\sigma\mu}\frac{i\,\gamma^{\rho}\gamma^{\nu}}{2}+g^{\sigma\nu}\frac{i\,\gamma^{\rho}\gamma^{\mu}}{2}}-g^{\rho\mu}\frac{i\,\gamma^{\nu}\gamma^{\sigma}}{2}+g^{\rho\nu}\frac{i\,\gamma^{\mu}\gamma^{\sigma}}{2}\right)\end{aligned}$$

El resultado se asemeja bastante a lo que buscamos, pero los términos recuadrados aparecen cambiados de signo y con el orden de las matrices cambiado. Usando (1) recuperamos el orden adecuado:

$$\begin{aligned}-g^{\sigma\mu}\gamma^{\rho}\gamma^{\nu}+g^{\sigma\nu}\gamma^{\rho}\gamma^{\mu} &= -g^{\sigma\mu}(2g^{\rho\sigma}-\gamma^{\nu}\gamma^{\rho})+g^{\sigma\nu}(2g^{\rho\mu}-\gamma^{\mu}\gamma^{\rho}) \\ &= \underbrace{-2g^{\sigma\mu}g^{\rho\sigma}+2g^{\sigma\nu}g^{\rho\mu}}_{0}+g^{\sigma\mu}\gamma^{\nu}\gamma^{\rho}-g^{\sigma\nu}\gamma^{\mu}\gamma^{\rho}\end{aligned}$$

Notemos que la única posibilidad de que los términos marcados sean diferentes de cero es $\mu=\nu=\rho=\sigma$; pero en ese caso tambien da cero al obtener 2 expresiones idénticas de signo opuesto.

2.- (b) Ver que

$$S[\Lambda_{rot(y)}]=cos\frac{\theta}{2}+\gamma^3\gamma^1 sin\frac{\theta}{2}$$

Operando análogamente al caso (a); tenemos

$$S[\Lambda]=exp\left(\frac{-i}{2}\omega_{\mu\nu}\frac{\sigma^{\mu\nu}}{2}\right)\ ;\ \sigma^{\mu\nu}=\frac{i}{2}[\gamma^{\mu},\gamma^{\nu}]$$

pero en este caso $\omega_{\mu\nu}=\theta$ y al tratarse de una rotación en torno al eje y, tomamos σ^{31}. Por tanto,

$$S[\Lambda_{rot(y)}]=exp\left(\frac{\theta}{2}(-i\sigma^{31})\right)=1+\frac{\theta}{2}(-i\sigma^{31})+\frac{1}{2!}\left(\frac{\theta}{2}(-i\sigma^{31})\right)^2+\frac{1}{3!}\left(\frac{\theta}{2}(-i\sigma^{31})\right)^3+\frac{1}{4!}\left(\frac{\theta}{2}(-i\sigma^{31})\right)^4+\cdots$$

Investiguemos el término $(-i\sigma^{31})^n$. Teniendo en cuenta que $\gamma^3\gamma^1=-\gamma^1\gamma^3$ y $\gamma^1\gamma^1=\gamma^3\gamma^3=-1$:

$$\begin{aligned}(-i\sigma^{31})^2 &= (-i)^2\sigma^{31}\sigma^{31}=(-i)^2\left(\frac{i}{2}[\gamma^3,\gamma^1]\right)^2=(-i)^2\left(\frac{i}{2}\right)^2(\gamma^3\gamma^1-\gamma^1\gamma^3)^2=\frac{-1}{4}(2\,\gamma^3\gamma^1)^2=-\gamma^3\gamma^1\gamma^3\gamma^1 \\ &= -\gamma^3\gamma^3\gamma^1\gamma^1=-1\end{aligned}$$

Así pués,

$$(-i\sigma^{31})^2 = -1$$
$$(-i\sigma^{31})^3 = (-i\sigma^{31})^2(-i\sigma^{31}) = -(-i\sigma^{31})$$
$$(-i\sigma^{31})^4 = (-i\sigma^{31})^2(-i\sigma^{31})^2 = 1$$
$$\vdots$$

Por tanto, el desarrollo en Taylos de la exponencial queda

$$S[\Lambda_{rot(y)}] = exp\left(\frac{\theta}{2}(-i\sigma^{31})\right) = 1 + \frac{\theta}{2}(-i\sigma^{31}) + \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 \underbrace{(-i\sigma^{31})^2}_{-1} + \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 \underbrace{(-i\sigma^{31})^3}_{-(-i\sigma^{31})} + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 \underbrace{(-i\sigma^{31})^4}_{1} + \cdots$$
$$= \left[1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 + \cdots\right] + (-i\sigma^{31})\left[\frac{\theta}{2} - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 (-i\sigma^{31}) + \cdots\right] = cos\frac{\theta}{2} + (-i\sigma^{31})sin\frac{\theta}{2}$$

Finalmente, reexpresando la σ en función de las γ's:

$$-i\,\sigma^{31} = (-i)\,\frac{i}{2}[\gamma^3,\gamma^1] = (-i)\frac{i}{2}\,(2\gamma^3\gamma^1) = \gamma^3\gamma^1$$

la expresión queda

$$S[\Lambda_{rot(y)}] = cos\frac{\theta}{2} + \gamma^3\gamma^1 sin\frac{\theta}{2} \quad \text{qed☺}$$

2.- (c) Ver que

$$S[\Lambda_{rot(z)}] = cos\frac{\phi}{2} + \gamma^1\gamma^2 sin\frac{\phi}{2}$$

Se procede igual que en el apartado (b) solo que ahora tomaremos $\omega_{\mu\nu} = \phi$ y $\sigma^{12} = \frac{i}{2}[\gamma^1,\gamma^2]$.